J. Fluid Mech. (1985), *vol.* 151, *pp.* 295–309
Printed in Great Britain

Exact evaluation of the effect of an arbitrary mean flow in kinematic dynamo theory

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(Received 7 November 1983 and in revised form 3 September 1984)

An exact derivation is given of the dynamo equation in the presence of an arbitrary incompressible mean flow $\boldsymbol{v}_0$. A mixed representation is used specifying $\boldsymbol{v}_0$ in terms of a Lagrangian displacement, while Eulerian coordinates are employed for the turbulent velocity $\boldsymbol{v}$ superimposed on $\boldsymbol{v}_0$.

When the first-order-smoothing approximation is made (FOSA; valid when the turbulent velocity $\boldsymbol{v}$ has a short correlation time τ_c) the usual dynamo equation is recovered, except that the turbulent velocity $\boldsymbol{v}$ in the tensors α_{is} and β_{isk} is replaced by $\bar{\boldsymbol{v}}$. The bar represents the effect of advection and is expressed solely in terms of the Lagrangian coordinate specifying the mean flow $\boldsymbol{v}_0$. Thus the intuitive idea is confirmed that dynamo action depends only on velocity correlation functions measured at a point comoving with the mean flow. The result admits easy evaluation in actual model situations. This is illustrated with an example tailored to the solar dynamo. A shear in $\boldsymbol{v}_0$ causes a (kinematic) anisotropy in the tensors α_{is} and β_{isk}. This can be a large effect, which comes on top of the intrinsic (dynamical) anisotropy in the velocity correlation functions. Subsequently, the analysis is extended beyond FOSA up to arbitrary order, relevant for long correlation times τ_c on the basis of the work of Van Kampen (1974). It is shown that the same formalism is also applicable to the problem of turbulent transport of a scalar.

Conditions for applicability of the work are (1) very large magnetic Reynolds number, (2) incompressible flows $\boldsymbol{v}$ and $\boldsymbol{v}_0$, (3) stationary mean flow $\boldsymbol{v}_0$, and (4) correlation time $\tau_c \ll$ period of the dynamo.

1. Introduction

The starting point of kinematic dynamo theory is the induction equation

$$\frac{\partial}{\partial t}\boldsymbol{B} = \nabla \times (\boldsymbol{V} \times \boldsymbol{B}). \tag{1}$$

The resistive term is omitted because attention is focused on the case of very high magnetic Reynolds number. The fluid motion $\boldsymbol{V}$ is regarded as incompressible ($\nabla \cdot \boldsymbol{V} = 0$) and is split into two parts: the (non-uniform) mean flow $\boldsymbol{v}_0$, constant in time, and the (stochastic) turbulent convection $\boldsymbol{v}$. Likewise, $\boldsymbol{B}$ is split into an average field $\boldsymbol{B}_0$ (the 'dynamo field') and a stochastic component $\boldsymbol{b}$. As usual, $\langle\ \rangle$ represents the ensemble average over all realizations of $\boldsymbol{v}$:

$$\boldsymbol{V} = \boldsymbol{v}_0 + \boldsymbol{v}, \quad \langle \boldsymbol{V} \rangle = \boldsymbol{v}_0, \quad \langle \boldsymbol{v} \rangle = 0, \tag{2a}$$

$$\boldsymbol{B} = \boldsymbol{B}_0 + \boldsymbol{b}, \quad \langle \boldsymbol{B} \rangle = \boldsymbol{B}_0, \quad \langle \boldsymbol{b} \rangle = 0, \tag{2b}$$

$$\nabla \cdot \boldsymbol{v}_0 = \nabla \cdot \boldsymbol{v} = 0, \quad \nabla \cdot \boldsymbol{B}_0 = \nabla \cdot \boldsymbol{b} = 0. \tag{2c}$$

By taking the ensemble average of (1), one obtains the following well-known equations for $\boldsymbol{B}_0$ and $\boldsymbol{b}$:

$$\frac{\partial}{\partial t}\boldsymbol{B}_0 = \nabla\times(\boldsymbol{v}_0\times\boldsymbol{B}_0+\langle\boldsymbol{v}\times\boldsymbol{b}\rangle), \tag{3a}$$

$$\frac{\partial}{\partial t}\boldsymbol{b} = \nabla\times(\boldsymbol{v}_0\times\boldsymbol{b}+\boldsymbol{v}\times\boldsymbol{B}_0+\boldsymbol{v}\times\boldsymbol{b}-\langle\boldsymbol{v}\times\boldsymbol{b}\rangle). \tag{3b}$$

At present, these equations can only be solved approximately. The usual approach is to ignore in ($3b$) the term $\boldsymbol{v}\times\boldsymbol{b}-\langle\boldsymbol{v}\times\boldsymbol{b}\rangle$. This is called the first-order-smoothing approximation (FOSA), which is justified when

$$\tau_c v/l \ll 1. \tag{4}$$

τ_c, v and l are the correlation time, the typical velocity and typical lengthscale associated with the turbulent velocity field $\boldsymbol{v}$. For physically interesting cases, (4) is likely to be violated. In the solar dynamo, $\tau_c v/l \sim 1$, so that FOSA is a bad approximation, but it is done nevertheless to keep the problem tractable. Subsequently, one also ignores the mean-flow term $\boldsymbol{v}_0\times\boldsymbol{b}$ in ($3b$), which is regarded as 'reasonable' (Cowling 1981). Then ($3b$) reduces to

$$\frac{\partial}{\partial t}\boldsymbol{b} = \nabla\times(\boldsymbol{v}\times\boldsymbol{B}_0), \tag{5}$$

which can be solved easily. Hence $\langle\boldsymbol{v}\times\boldsymbol{b}\rangle$ can be expressed in terms of $\boldsymbol{B}_0$, which is needed in ($3a$) to obtain a closed equation for $\boldsymbol{B}_0$, the dynamo equation.

The FOSA approach can be avoided, and an exact equation for $\boldsymbol{B}_0$ can be found from ($3a, b$) by applying the theory of stochastic differential equations (Van Kampen 1974). The result involves an infinite series containing progressively higher-order velocity correlations, which for $\tau_c v/l \sim 1$ converges slowly, if at all. Knobloch (1977, 1978) has evaluated the first few terms, but only for $\boldsymbol{v}_0 = 0$.

The case of a non-vanishing mean flow has been studied by Krause & Rädler (1971, 1980, chap. 8) and by Krause (1973, 1976). These authors considered a mean flow with *constant* shear and a non-zero molecular resistivity. Rädler (1980) discusses the more general case of a non-uniform rotation, in combination with a compressible turbulent velocity $\boldsymbol{v}$.

The drawback of these existing treatments is that their results are usually rather complicated and not easy to interpret physically. My object is to show that the problem of turbulent transport of a vector $\boldsymbol{B}$ at arbitrary mean flow and zero resistivity is amenable to a general and systematic treatment, leading to an intuitively simple result, which can be extended beyond FOSA to arbitrary order, and to turbulent transport of a scalar.

This article evaluates the *kinematic* effects of the $\boldsymbol{v}_0\times\boldsymbol{b}$ term in ($3b$) in terms of velocity correlation functions. Specific tensorial properties of the latter are not considered here, but these have investigated extensively (see e.g. Moffatt 1978; Krause & Rädler 1980).

2. Operator notation

It is profitable to employ a formal notation in terms of two operators R and C:

$$\mathsf{R} \equiv \nabla\times(\boldsymbol{v}_0\times\quad), \quad \mathsf{C} \equiv \nabla\times(\boldsymbol{v}\times\quad). \tag{6}$$

$\boldsymbol{v}_0$ contains all systematic, non-stochastic fluid motion. There are no restrictions on the nature of $\boldsymbol{v}_0$ other than that $\boldsymbol{v}_0$ should be time-independent and $\nabla\cdot\boldsymbol{v}_0 = 0$.

With these definitions

$$(1) \to \frac{\partial}{\partial t}\boldsymbol{B} = (\mathbf{R}+\mathbf{C})\,\boldsymbol{B}, \tag{7}$$

and, using $\langle \mathbf{R} \rangle = \mathbf{R}$ and $\langle \mathbf{C} \rangle = 0$,

$$(3a) \to \frac{\partial}{\partial t}\boldsymbol{B}_0 = \mathbf{R}\boldsymbol{B}_0 + \langle \mathbf{C}\boldsymbol{b} \rangle \tag{8a}$$

$$(3b) \to \frac{\partial}{\partial t}\boldsymbol{b} = \mathbf{R}\boldsymbol{b} + \mathbf{C}\boldsymbol{B}_0 + \mathbf{C}\boldsymbol{b} - \langle \mathbf{C}\boldsymbol{b} \rangle. \tag{8b}$$

Now, we ignore the last two terms in (8*b*), i.e. FOSA, and substitute

$$\boldsymbol{b} = \mathrm{e}^{t\mathbf{R}}\boldsymbol{\beta} \tag{9}$$

in the remainder. Transformation (9) is applied frequently here, and exclusively to vector fields having zero divergence. Here we merely note that $\exp(-t\mathbf{R})$ is the inverse of $\exp(t\mathbf{R})$ and that $\exp(t\mathbf{R})$ conserves the value of the divergence. Both assertions follow directly from the exponential series expansion. For example, for any vector $\boldsymbol{a}$,

$$\nabla \cdot \{\mathrm{e}^{t\mathbf{R}}\boldsymbol{a}\} = \nabla \cdot \Sigma \frac{1}{n!}(t\mathbf{R})^n \boldsymbol{a} = \nabla \cdot \boldsymbol{a}, \tag{10}$$

because $\nabla \cdot \mathbf{R} = 0$. Since $\nabla \cdot \boldsymbol{b} = 0$, it follows that $\nabla \cdot \boldsymbol{\beta} = 0$.

For $\boldsymbol{\beta}$ one obtains the equation

$$\frac{\partial}{\partial t}\boldsymbol{\beta} = \mathrm{e}^{-t\mathbf{R}}\,\mathbf{C}\boldsymbol{B}_0, \tag{11}$$

which can be solved immediately:

$$\boldsymbol{\beta}(t) = \int_{-\infty}^{t} \mathrm{d}s\, \mathrm{e}^{-s\mathbf{R}}\,\mathbf{C}(s)\,\boldsymbol{B}_0(s) \;+\; \boldsymbol{\beta}(-\infty). \tag{12}$$

At this point, the convention is adopted that the explicit position and time argument are always $\boldsymbol{r}$ *and* t, *respectively, unless otherwise indicated.* For example

$$\boldsymbol{B}_0 = \boldsymbol{B}_0(t) = \boldsymbol{B}_0(\boldsymbol{r},t), \quad \mathbf{C}(s) = \nabla \times (\boldsymbol{v}(\boldsymbol{r},s) \times \;\;), \quad \mathbf{C} = \nabla \times (\boldsymbol{v}(\boldsymbol{r},t) \times \;\;), \quad \text{etc.}$$

Putting $s = t-\tau$, it follows from (12) that

$$\langle \mathbf{C}\boldsymbol{b} \rangle = \langle \mathbf{C}\,\mathrm{e}^{t\mathbf{R}}\boldsymbol{\beta} \rangle = \int_0^{\infty} \mathrm{d}\tau\, \langle \mathbf{C}(t)\,\mathrm{e}^{\tau\mathbf{R}}\,\mathbf{C}(t-\tau) \rangle\, \boldsymbol{B}_0(t-\tau). \tag{13}$$

The contribution of $\boldsymbol{\beta}(-\infty)$ vanishes since there is no correlation between $\boldsymbol{v}(t)$ and $\boldsymbol{b}(t_0)$ as soon as $|t-t_0| \gg \tau_c$. One may now substitute (13) in (8*a*). To obtain a closed equation for $\boldsymbol{B}_0$, one must express $\boldsymbol{B}_0(t-\tau)$ in (13) in terms of $\boldsymbol{B}_0 \equiv \boldsymbol{B}_0(\boldsymbol{r},t)$. The theory of stochastic differential equations (Van Kampen 1976, §12, 1981, chap. 14) shows that for a short correlation time (i.e. when (4) holds) the result is

$$\frac{\partial}{\partial t}\boldsymbol{B}_0 = \left\{\mathbf{R} + \int_0^{\infty} \mathrm{d}\tau\, \langle \mathbf{C}(t)\,\mathrm{e}^{\tau\mathbf{R}}\,\mathbf{C}(t-\tau)\,\mathrm{e}^{-\tau\mathbf{R}} \rangle\right\}\boldsymbol{B}_0. \tag{14}$$

This is the dynamo equation in disguise. In retrospect, it is seen that (14) implies the substitution $\boldsymbol{B}_0(t-\tau) = \exp(-\tau\mathbf{R})\,\boldsymbol{B}_0$ in (13), or, explicitly,

$$\boldsymbol{B}_0(\boldsymbol{r},t) = \mathrm{e}^{\tau\mathbf{R}}\,\boldsymbol{B}_0(\boldsymbol{r},t-\tau) \quad \text{for } \tau \lesssim \tau_c. \tag{15}$$

Relation (15) need only hold for $\tau \lesssim \tau_c$, as the correlation function $\langle\ \rangle$ in (13) vanishes for $\tau \gg \tau_c$ anyway. Note that (14), and therefore also (15), emerges from the theory, where no other assumption than (4) is made: relation (14), and therefore (15), guarantees that *all* terms up to order $(v\tau_c/l)^2$ are accounted for. Apparently, one may ignore the stochastic term $\langle \mathbf{C}\boldsymbol{b}\rangle$ in (8*a*) for short times ($\tau < \tau_c$). Integration then leads to (15). Relation (15) is also the proper formulation of the usual statement that 'the dynamo field changes only slowly in time'. This matter is discussed further in §4.

3. The case of zero mean flow

To make contact with known results, suppose one replaces in (14) $\exp(\pm t\mathbf{R}) \rightarrow 1$, thus ignoring all effects of rotation and shear induced by a non-zero mean flow. Then (14) reads

$$\frac{\partial}{\partial t}\boldsymbol{B}_0 = \left\{\mathbf{R} + \int_0^\infty \mathrm{d}\tau \,\langle \mathbf{C}(t)\,\mathbf{C}(t-\tau)\rangle\right\}\boldsymbol{B}_0.$$

Writing out $\mathbf{R}$ and $\mathbf{C}$ explicitly, this is seen to be equivalent to

$$\frac{\partial}{\partial t}\boldsymbol{B}_0 = \boldsymbol{\nabla}\times(\boldsymbol{v}_0\times\boldsymbol{B}_0+\boldsymbol{\mathscr{E}}), \tag{16a}$$

$$\boldsymbol{\mathscr{E}} = \int_0^\infty \mathrm{d}\tau\,\langle \boldsymbol{v}(t)\times\boldsymbol{\nabla}\times\{\boldsymbol{v}(t-\tau)\times\boldsymbol{B}_0\}\rangle. \tag{16b}$$

Exactly the same is found by solving $\boldsymbol{b}$ from (5) and then computing $\langle \boldsymbol{v}\times\boldsymbol{b}\rangle$. Working out the vector products with $\boldsymbol{\nabla}\cdot\boldsymbol{v} = \boldsymbol{\nabla}\cdot\boldsymbol{B}_0 = 0$ gives

$$\mathscr{E}_i = \alpha_{is}B_{0,s} - \beta_{isk}\nabla_s B_{0,k}, \tag{17}$$

with

$$\alpha_{is} = \epsilon_{ijk}\int_0^\infty \mathrm{d}\tau\,\langle v_j(t)\,\nabla_s v_k(t-\tau)\rangle, \tag{18a}$$

$$\beta_{isk} = \epsilon_{ijk}\int_0^\infty \mathrm{d}\tau\,\langle v_j(t)\,v_s(t-\tau)\rangle. \tag{18b}$$

Substitution of (17) in (16*a*) yields the dynamo equation in its usual form. Further simplification is frequently sought by supposing that the turbulent velocity $\boldsymbol{v}$ in (18*a*, *b*) is isotropic.

The following conceptual difficulty arises of one applies (18) to compute $\alpha_{is}(\boldsymbol{r},t)$ and $\beta_{isk}(\boldsymbol{r},t)$ when $\boldsymbol{v}_0 \neq 0$, as is commonly done. The correlation functions in (18) contain $\boldsymbol{v}(\boldsymbol{r},t)$ and $\boldsymbol{v}(\boldsymbol{r},t-\tau)$. These velocities are measured at the same fixed position $\boldsymbol{r}$, but at quite different *material points* in the dynamo (see figure 1). Consider a fluid element that is at $\boldsymbol{r}$ at time $t-\tau$, when its velocity is measured to find $\boldsymbol{v}(\boldsymbol{r},t-\tau)$. After τ seconds this experiment is repeated to obtain $\boldsymbol{v}(\boldsymbol{r},t)$, but the original fluid element has moved to B. From a statistical point of view, position B is uncertain to within a sphere (say) with centre at A′ found by following the mean flow for τ seconds. The linear distance AB can be estimated by AA′, and this can be much larger than a correlation length l, even if $\tau \lesssim \tau_c$. For $\tau \sim \tau_c$ one finds $\mathrm{AA}'/l \sim 5$ for the upper convection zone of the Sun, and much for rapidly rotating stars.

One could argue that there will be no velocity correlation between the material positions A and B since $\mathrm{AB} \gg l$, so that the contribution to the integrals in (18) is simply zero. The implication is that α_{is} and β_{isk} defined by (18) become small and eventually approach zero for rapid rotators. However, it seems intuitively obvious that dynamo action at a given point (and therefore the tensors α_{is} and β_{isk}) may only

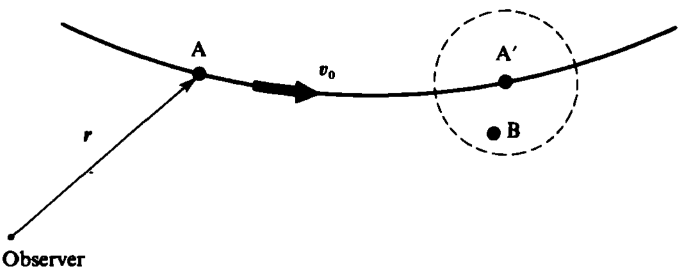

FIGURE 1. A material point that is at A at time $t-\tau$ will have moved to B after a time interval τ.

depend on velocity correlations measured at this point *as one moves along with it at the mean fluid speed* $\boldsymbol{v}_0$. This is just a matter of reference frames: if one observes from a moving frame that such $\boldsymbol{v}_0$ vanishes locally, say at point A, then one naturally expects dynamo action at A to depend only on what happens in the (stationary) point A. We shall prove that this is actually the case. In doing so, not only this effect of $\boldsymbol{v}_0$ itself is recovered; the derivatives of $\boldsymbol{v}_0$ (the shear) turn out to have a large influence on α_{is} and β_{isk} as well.

4. Analysis of the mean flow-operator exp $(\tau\mathbf{R})$

In order to proceed with (14), the meaning of exp $(\tau\mathbf{R})$ must be established. This is straightforward on the basis of techniques available in the literature. The developments below up to relation (27) are to be regarded as a definition of notation. Use will be made of a mixed Eulerian–Lagrangian representation. This technique was introduced by Soward (1972), and generalized by Andrews & McIntyre (1978). These authors envisage a mean flow on which the fluctuations are represented by Lagrangian displacements (see in particular Moffat 1978, §§8.2–8.5). We shall take the opposite approach: the mean flow $\boldsymbol{v}_0$ is treated in terms of a Lagrangian coordinate, whereas the Eulerian representation is used for the turbulent velocity $\boldsymbol{v}$ superimposed on $\boldsymbol{v}_0$.

Let $\boldsymbol{a}$ be a divergence-free vector field, and define

$$\boldsymbol{u}(\boldsymbol{r},t) \equiv \mathrm{e}^{t\mathbf{R}}\,\boldsymbol{a}(\boldsymbol{r},t_0). \tag{19}$$

The vector $\boldsymbol{u}$ satisfies the equation

$$\frac{\partial}{\partial t}\boldsymbol{u} = \mathbf{R}\boldsymbol{u} = \nabla\times(\boldsymbol{v}_0\times\boldsymbol{u}), \tag{20}$$

which means that $\boldsymbol{u}$ is *advected by* the mean flow $\boldsymbol{v}_0$. For incompressible $\boldsymbol{v}_0$, (20) has the following Lagrangian solution (Roberts 1967, §2.3):

$$\boldsymbol{u}(\boldsymbol{r},t) = \boldsymbol{D}^{-t}\,\boldsymbol{u}(\boldsymbol{r}^{-t},0), \tag{21}$$

with

$$D^t_{ij} \equiv \frac{\partial x_i}{\partial x^t_j} \quad \text{or} \quad \boldsymbol{D}^t = \frac{\partial \boldsymbol{r}}{\partial \boldsymbol{r}^t} \tag{22}$$

(Roberts 1967, §1.7). The vector $\boldsymbol{r}$ is a fixed (Eulerian) position, and $\boldsymbol{r}^t$ is the (Lagrangian) position of an imaginary material point moving with the mean flow $\boldsymbol{v}_0$ t seconds after its position was $\boldsymbol{r}$. The following relation is an elementary consequence of the chain rule:

$$(D^t)^{-1}{}_{ij} = \frac{\partial x^t_i}{\partial x_j} \quad \text{or} \quad (\boldsymbol{D}^t)^{-1} = \frac{\partial \boldsymbol{r}^t}{\partial \boldsymbol{r}}. \tag{23}$$

If we allow first a time p to elapse and then another q, we obtain the identity $(\boldsymbol{r}^p)^q = \boldsymbol{r}^{p+q}$. In particular, $(\boldsymbol{r}^{-t})^t = \boldsymbol{r}^0 = \boldsymbol{r}$. The upper index to $\boldsymbol{r}$ indicates the *elapsed* time. Such a construction is possible since $\boldsymbol{v}_0$ is stationary. The substitution $\boldsymbol{r} \to \boldsymbol{r}^{-t}$ changes $\boldsymbol{D}^t$ into $\partial \boldsymbol{r}^{-t}/\partial(\boldsymbol{r}^{-t})^t = \partial \boldsymbol{r}^{-t}/\partial \boldsymbol{r} = (\boldsymbol{D}^{-t})^{-1}$:

$$\boldsymbol{D}^t \xrightarrow{\boldsymbol{r}\to\boldsymbol{r}^{-t}} (\boldsymbol{D}^{-t})^{-1}. \tag{24}$$

$\boldsymbol{D}^t$ and their inverses equal the unit matrix $\boldsymbol{I}$ at $t = 0$. From $\nabla\cdot\boldsymbol{v}_0 = 0$ it follows that (Roberts 1967, §1.7)

$$\det \boldsymbol{D}^t = 1. \tag{25}$$

Relation (21) may be written in terms of the vector field $\boldsymbol{a}$:

$$e^{t\mathbf{R}}\, \boldsymbol{a}(\boldsymbol{r}, t_0) = \boldsymbol{D}^{-t}\, \boldsymbol{a}(\boldsymbol{r}^{-t}, t_0). \tag{26}$$

Pictorially, $\exp(t\mathbf{R})\, \boldsymbol{a}(\boldsymbol{r}, t_0)$ is generated by letting passive advection by the mean flow operate during a time interval t on a snapshot of the vector field $\boldsymbol{a}(\boldsymbol{r}, t)$ taken at $t = t_0$. The result is not equal to $\boldsymbol{a}(\boldsymbol{r}, t_0 + t)$, since $\boldsymbol{a}$ is supposed to evolve in time independently. Only if $\boldsymbol{a}$ itself is advected by the mean flow, i.e. if $\boldsymbol{a}$ also satisfies (20), may we conclude that

$$\boldsymbol{a}(\boldsymbol{r}, t_0 + t) = e^{t\mathbf{R}}\, \boldsymbol{a}(\boldsymbol{r}, t_0) = \boldsymbol{D}^{-t}\, \boldsymbol{a}(\boldsymbol{r}^{-t}, t_0). \tag{27}$$

Before turning to the analysis of (14), the meaning of (15) is discussed. One might think that the usual statement that 'the dynamo field $\boldsymbol{B}_0$ varies only slowly in time' justifies the approximation $\boldsymbol{B}_0(t-\tau) \approx \boldsymbol{B}_0(t) \equiv \boldsymbol{B}_0$ in (13). This would be correct in the case of an axisymmetric dynamo, but in general it is not. For instance, if one visualizes a non-axisymmetric dynamo in a rapidly rotating star, $\boldsymbol{B}_0(\boldsymbol{r}, t)$ may be quite different from $\boldsymbol{B}_0(\boldsymbol{r}, t-\tau)$, as the mean flow, in this case a rotation, may move a completely different value of $\boldsymbol{B}_0$ to the *fixed* position $\boldsymbol{r}$. A more proper statement would therefore be that, for timescales τ of the order of τ_c, the dynamo field $\boldsymbol{B}_0$ experiences negligible intrinsic time evolution *other than* passive advection by $\boldsymbol{v}_0$. This is certainly reasonable if τ_c is much smaller than the period of the dynamo. The implication is that, for $\tau \lesssim \tau_c$, $\boldsymbol{B}_0$ is advected so that (27) applies. Taking $t = \tau$ and $t_0 = t - \tau$ there, one recovers (15). One thing must be emphasized now: (15) is not an assumption, but a *consequence* of the theory of stochastic differential equations, which in retrospect may be interpreted by saying that $\boldsymbol{B}_0$ is passively advected by the mean flow for $\tau \lesssim \tau_c$.

It remains to evaluate the transformation $\exp(\tau\mathbf{R})\, \mathbf{C}(t-\tau) \exp(-\tau\mathbf{R})$ in (14). Consider a divergence-free vector field $\boldsymbol{a}$ (not advected). Then

$$\begin{aligned}\mathbf{O}\boldsymbol{a} &\equiv e^{\tau\mathbf{R}}\, \mathbf{C}(t-\tau)\, e^{-\tau\mathbf{R}}\, \boldsymbol{a} \\ &= e^{\tau\mathbf{R}}\, \nabla \times [\boldsymbol{v}(\boldsymbol{r}, t-\tau) \times e^{-\tau\mathbf{R}}\, \boldsymbol{a}(\boldsymbol{r}, t)] \\ &= e^{\tau\mathbf{R}}\, \nabla \times [\boldsymbol{v}(\boldsymbol{r}, t-\tau) \times \{\boldsymbol{D}^{\tau}\, \boldsymbol{a}(\boldsymbol{r}^{\tau}, t)\}].\end{aligned} \tag{28}$$

The second line just writes $\mathbf{C}(t-\tau)$ explicitly and the third is an application of (26).

According to (26), $\exp(\tau\mathbf{R})$ in (28) is equivalent to a $\boldsymbol{D}^{-\tau}$ in front and substitution of $\boldsymbol{r} \to \boldsymbol{r}^{-\tau}$ *everywhere* else. In particular, $\boldsymbol{D}^{\tau}$ changes into $(\boldsymbol{D}^{-\tau})^{-1}$, see (24):

$$\mathbf{O}\boldsymbol{a} = \boldsymbol{D}^{-\tau}[\nabla^{-\tau} \times \{\boldsymbol{v}(\boldsymbol{r}^{-\tau}, t-\tau) \times (\boldsymbol{D}^{-\tau})^{-1}\, \boldsymbol{a}(\boldsymbol{r}, t)\}], \tag{29}$$

where ${\nabla^{-\tau}}_i = \partial/\partial {x^{-\tau}}_i$. To establish how $\boldsymbol{D}^{-\tau}$ acts on the double outer product in (29)

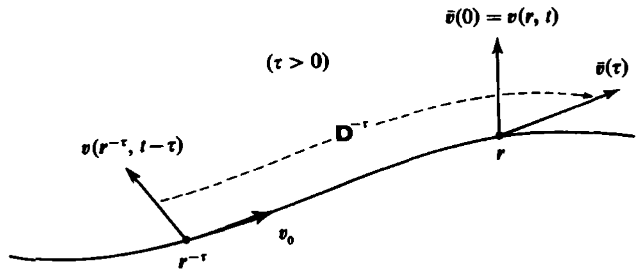

FIGURE 2. The vectors $\bar{v}(0)$ and $\bar{v}(\tau)$ occurring in (37*a*, *b*). The trajectory is shown of an (imaginary) material point moving with the mean flow, whose position is r at time t. The turbulent velocity v is measured (in a frame moving with the mean flow, so that $\langle v \rangle = 0$) at the position $r^{-\tau}$ of that point at time $t-\tau$. The resulting vector is then allowed to be passively advected by the mean flow for a time interval τ to position r and time t; this is done by applying the matrix $D^{-\tau}$. In this way an intrinsic (kinematic) anisotropy is generated.

is somewhat technical and therefore deferred to Appendix A. The result is simple:

$$\begin{aligned} \mathbf{O}a &= \nabla \times [\{D^{-\tau} v(r^{-\tau}, t-\tau\} \times a(r,t)] \\ &= \nabla \times [\{e^{\tau \mathbf{R}} v(r, t-\tau)\} \times a(r,t)]. \end{aligned} \tag{30}$$

In the last line, we have again used (26). We now define the 'velocity' $\bar{v}$ in progressive shorthand notation

$$\bar{v}(\tau) \equiv D^{-\tau} v(r^{-\tau}, t-\tau) = e^{\tau \mathbf{R}} v(r, t-\tau) = e^{\tau \mathbf{R}} v(t-\tau). \tag{31}$$

$\bar{v}$ has really three arguments, $\bar{v} = \bar{v}(r, t, \tau)$, but, to keep the notation flexible and conform to the shorthand convention adopted in §2, we shall take the first two for granted, see also figure 2. Then (28), (30) and (31) imply

$$\begin{aligned} e^{\tau \mathbf{R}} \nabla \times [v(r, t-\tau) \times e^{-\tau \mathbf{R}} \quad] &= \nabla \times [\{e^{\tau \mathbf{R}} v(r, t-\tau)\} \times \quad] \\ &= \nabla \times [\bar{v}(\tau) \times \quad]. \end{aligned} \tag{32}$$

Equation (32) provides the basis for the results derived in this paper. More succinctly, for application to divergence-free vector fields,

$$e^{\tau \mathbf{R}} \mathbf{C}(t-\tau) e^{-\tau \mathbf{R}} = \bar{\mathbf{C}}(\tau), \quad \bar{\mathbf{C}}(\tau) \equiv \nabla \times [\bar{v}(\tau) \times \quad]. \tag{33}$$

$\bar{\mathbf{C}}$ is defined as in (6), but now with $\bar{v}$ instead of v. Note, finally, the relation

$$\bar{\mathbf{C}}(0) = \nabla \times [v(r,t) \times \quad] = \mathbf{C}(t) = \mathbf{C}, \tag{34}$$

since $\bar{v}(0) = v(r,t) = v(t) = v$, in shorthand notation.

5. Exact results within the first-order-smoothing approximation

With the last two relations (33) and (34), one may write (14) as

$$\frac{\partial}{\partial t} B_0 = \left\{ \mathbf{R} + \int_0^\infty d\tau \langle \bar{\mathbf{C}}(0) \bar{\mathbf{C}}(\tau) \rangle \right\} B_0, \tag{35}$$

or, writing $\mathbf{R}$ and $\bar{\mathbf{C}}$ explicitly,

$$\frac{\partial}{\partial t} B_0 = \nabla \times (v_0 \times B_0 + \mathscr{E}), \quad \mathscr{E}_i = \alpha_{is} B_{0,s} - \beta_{isk} \nabla_s B_{0,k}, \tag{36a, b}$$

with

$$\alpha_{is} = \epsilon_{ijk} \int_0^\infty \mathrm{d}\tau \, \langle \bar{v}_j(0) \, \nabla_s \bar{v}_k(\tau) \rangle, \quad \beta_{isk} = \epsilon_{ijk} \int_0^\infty \mathrm{d}\tau \, \langle \bar{v}_j(0) \, \bar{v}_s(\tau) \rangle. \qquad (37\,a, b)$$

These exact results look very similar to the usual approximate relations (16)–(18). The only difference brought about by incorporation of the $\boldsymbol{v}_0 \times \boldsymbol{b}$ term from (3*b*) in the theory is that $\bar{\boldsymbol{v}}$ appears in (37) instead of $\boldsymbol{v}$. In §7 this is proved to hold not only within FOSA, but in all higher-order approximations as well.

The meaning of (37*a*, *b*) is explained in figure 2, and it is clear that the tensors $\alpha_{is}(\boldsymbol{r}, t)$ and $\beta_{isk}(\boldsymbol{r}, t)$ defined by (37*a*, *b*) depend only on the turbulent velocity $\boldsymbol{v}$ measured at the position of *one* material point moving with the mean flow such that its position is $\boldsymbol{r}$ at time t. This confirms the point made at the end of §3. The passive advection applied to $\boldsymbol{v}(\boldsymbol{r}^{-t}, t-\tau)$ to generate the vector $\bar{\boldsymbol{v}}(\tau)$ is the origin of an additional anisotropy in the tensors α_{is} and β_{isk}. Relation (37*a*,*b*) is not only very simple, but also straightforward to evaluate, as the example of §6 shows. The interested reader is referred to Appendix B, where it is shown that the problem of turbulent transport of a scalar (instead of a vector as above) at non-zero mean flow can be dealt with in the same way.

6. Example

The advantage of the formal expressions (37*a*, *b*) is that they can be evaluated once the Lagrangian coordinate $\boldsymbol{r}^t$ for the mean flow is specified. In this section we shall follow the simplified notation defined in figure 3, taking $t = \tau$ there. We choose for $\boldsymbol{v}_0$ a rotation, $\boldsymbol{v}_0 = \boldsymbol{\Omega} \times \boldsymbol{r}$, with non-uniform $\boldsymbol{\Omega}$ to be specified later. In this case it is useful to transform to a frame that is locally rigidly corotating: let $\mathcal{D}$ be the rigid rotation part of $\boldsymbol{D}$, i.e. what remains of $\boldsymbol{D}$ if $\nabla \Omega$ is put equal to zero. $\mathcal{D}$ being orthogonal, we write (31) as

$$\bar{\boldsymbol{v}}(\tau) = \boldsymbol{D}\mathcal{D}^*\mathcal{D}\boldsymbol{v}(\boldsymbol{r}^{-\tau}, t-\tau) \equiv \boldsymbol{\Delta}\tilde{\boldsymbol{v}}(\tau), \qquad (38)$$

where $*$ indicates the transposed matrix, and

$$\boldsymbol{\Delta} = \boldsymbol{D}\mathcal{D}^*, \qquad (39)$$

$$\tilde{\boldsymbol{v}}(\tau) = \mathcal{D}\boldsymbol{v}(\boldsymbol{r}^{-\tau}, t-\tau). \qquad (40)$$

Note the difference between $\bar{\boldsymbol{v}}$ and $\tilde{\boldsymbol{v}}$. Rädler (1980) uses transformation (40) – a quasi-rigid rotation at the local value of $\boldsymbol{\Omega}$ – in his analysis. Note also that $\nabla \cdot \tilde{\boldsymbol{v}}(\tau) \neq 0$, whereas $\nabla \cdot \bar{\boldsymbol{v}}(\tau) = 0$.

With these definitions, (37*a*, *b*) become

$$\alpha_{is} = \epsilon_{ijk} \int_0^\infty \mathrm{d}\tau \, \Delta_{kl} \langle \tilde{v}_j(0) \, \nabla_s \tilde{v}_l(\tau) \rangle + \epsilon_{ijk} \int_0^\infty \mathrm{d}\tau \, (\nabla_s \Delta_{kl}) \, \langle \tilde{v}_j(0) \, \tilde{v}_l(\tau) \rangle, \qquad (41\,a)$$

$$\beta_{isk} = \epsilon_{ijk} \int_0^\infty \mathrm{d}\tau \, \Delta_{sl} \langle \tilde{v}_j(0) \, \tilde{v}_l(\tau) \rangle. \qquad (41\,b)$$

In (41) the (kinematic) distorting effects of the mean flow have been removed from the velocity correlation functions and put together in the coefficients Δ_{ij}. The velocities $\tilde{\boldsymbol{v}}$ may be thought of as turbulent velocities corrected for rigid rotation *without* shear. Note that α_{is} contains an extra contribution if there is shear ($\nabla_s \Delta_{kl} \neq 0$). This term appears to be diffusive in nature, as the velocity correlation

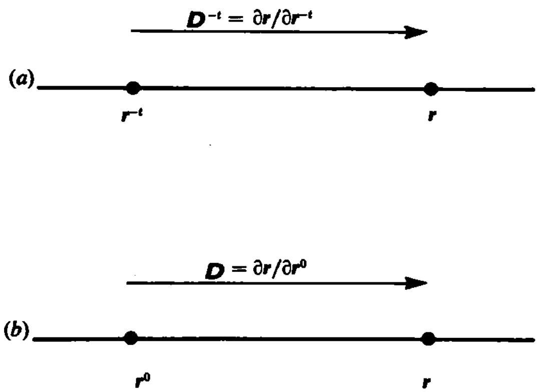

FIGURE 3. (*a*) Notation employed in §4. (*b*) Corresponding simplified notation used in §6 and Appendix A.

function contains no derivative. One might object that the effect appears solely as a result of transformation (39), but I shall refrain from a full discussion. Diffusive or not, this extra contribution to α_{is} is related to *turbulent* diffusion and therefore different from the diffusive α-effect in the work of Braginskii (1965*a*, *b*), which is proportional to the *molecular* resistivity.

To proceed with the evaluation of (41), we take the z-axis parallel to $\boldsymbol{\Omega}$, with Ω depending only on z, as a rough model of solar differential rotation (Stix 1981). The Lagrangian coordinate $\boldsymbol{r}(\boldsymbol{r}^0, \tau)$ of a material point carried by the mean flow $\boldsymbol{v}_0$ is then given by

$$x = r\cos(\Omega\tau + \phi^0), \quad y = r\sin(\Omega\tau+\phi^0), \quad z = z^0, \tag{42}$$

with $r^2 = (x^0)^2 + (y^0)^2 = x^2+y^2$, $\Omega = \Omega(z)$, $\phi^0 = \arctan(y^0/x^0)$, so that $\boldsymbol{r} = \boldsymbol{r}^0$ for $\tau = 0$. From (22) and figure 3 one obtains $D_{ij} = \partial x_i / \partial x^0{}_j$, or with (42),

$$\boldsymbol{D} = \frac{1}{r^2}\begin{pmatrix} xx^0+yy^0 & xy^0-yx^0 & 0 \\ yx^0-xy^0 & xx^0+yy^0 & 0 \\ 0 & 0 & r^2 \end{pmatrix} + \tau\frac{\partial\Omega}{\partial z}\begin{pmatrix} 0 & 0 & -y \\ 0 & 0 & x \\ 0 & 0 & 0 \end{pmatrix}. \tag{43}$$

The first matrix in (43) is $\mathscr{D}$, so that $\boldsymbol{\Delta}$ becomes

$$\boldsymbol{\Delta} = \boldsymbol{D}\mathscr{D}^* = \boldsymbol{I} + \tau\frac{\partial\Omega}{\partial z}\begin{pmatrix} 0 & 0 & -y \\ 0 & 0 & x \\ 0 & 0 & 0 \end{pmatrix}. \tag{44}$$

Further simplification is gained by transforming to cylindrical coordinates. For $\boldsymbol{\Delta}$ this is trivially achieved by taking $x = r$ and $y = 0$ in (44), since then the frames $(\boldsymbol{e}_x, \boldsymbol{e}_y, \boldsymbol{e}_z)$ and $(\boldsymbol{e}_r, \boldsymbol{e}_\phi, \boldsymbol{e}_z)$ coincide:

$$\boldsymbol{\Delta} = \boldsymbol{I} + \tau\Omega'\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \Omega' \pm \equiv r\frac{\partial\Omega}{\partial z}. \tag{45}$$

Owing to symmetry, (45) is generally valid. Hence

$$\Delta_{ij} = \delta_{ij} + \tau\Omega'\,\delta_{i2}\,\delta_{j3}, \tag{46}$$

with the convention $1 = r$, $2 = \phi$, $3 = z$. $\boldsymbol{e}_r$ and $\boldsymbol{e}_\phi$ corotate rigidly, but $\boldsymbol{e}_z$ is tilted when $\Omega' \neq 0$, as is confirmed by the following relations:

$$\Delta \boldsymbol{e}_r = \boldsymbol{e}_r, \quad \Delta \boldsymbol{e}_\phi = \boldsymbol{e}_\phi, \quad \Delta \boldsymbol{e}_z = \boldsymbol{e}_z + \tau\Omega' \boldsymbol{e}_\phi.$$

Furthermore, we take

$$\langle \tilde{v}_j(0)\, \nabla_s\, \tilde{v}_l(\tau)\rangle = \tfrac{1}{6} f(\tau)\, \epsilon_{jsl}, \tag{47a}$$

$$\langle \tilde{v}_j(0)\, \tilde{v}_l(\tau)\rangle = \tfrac{1}{3} g(\tau)\, \delta_{jl}, \tag{47b}$$

with
$$f(\tau) \equiv \langle \tilde{\boldsymbol{v}}(0)\cdot\nabla\times\tilde{\boldsymbol{v}}(\tau)\rangle, \quad g(\tau) \equiv \langle \tilde{\boldsymbol{v}}(0)\cdot\tilde{\boldsymbol{v}}(\tau)\rangle \tag{48}$$

(f and g may also depend on $\boldsymbol{r}$). This is the usual assumption of statistically isotropic turbulence. More precisely, it means that $\boldsymbol{v}$ after *correction for locally rigid rotation*, in other words $\tilde{\boldsymbol{v}}$, is statistically isotropic. This assumption is made here for simplicity and to single out the kinematic distorting effects. Of course, it is not realistic to ignore at the same time the anisotropy in the velocity correlations caused by dynamical effects (such as Coriolis forces; see further Moffatt 1978; Krause & Rädler 1980), but again this is done merely for the sake of a simple example.

The remaining calculation is standard. Defining the moments

$$\left.\begin{aligned} f_0 &\equiv \int_0^\infty \mathrm{d}\tau\, f(\tau), \quad f_1 \equiv \int_0^\infty \tau\, \mathrm{d}\tau\, f(\tau), \\ g_0 &\equiv \int_0^\infty \mathrm{d}\tau\, g(\tau), \quad g_1 \equiv \int_0^\infty \tau\, \mathrm{d}\tau\, g(\tau), \end{aligned}\right\} \tag{49}$$

we obtain, after inserting (46) and (47) in (41),

$$\mathscr{E} = \boldsymbol{\alpha}\cdot\boldsymbol{B}_0 - \tfrac{1}{3} g_0 \nabla\times\boldsymbol{B}_0 - \frac{1}{3}\frac{\partial\Omega}{\partial z}\frac{g_1}{r}\boldsymbol{e}_z\times(\boldsymbol{e}_\phi\cdot\nabla)\boldsymbol{B}_0, \tag{50}$$

with
$$\boldsymbol{\alpha} = \begin{pmatrix} \alpha_{rr} & 0 & \alpha_{rz} \\ 0 & \alpha_{\phi\phi} & 0 \\ 0 & \alpha_{z\phi} & \alpha_{zz} \end{pmatrix},$$

$$\alpha_{rr} = \alpha_{\phi\phi} = -\tfrac{1}{3} f_0 - \frac{1}{3}\frac{\partial\Omega}{\partial z} g_1, \quad \alpha_{zz} = -\tfrac{1}{3} f_0, \quad \alpha_{rz} = -\tfrac{1}{3} r \frac{\partial^2\Omega}{\partial z^2} g_1, \quad \alpha_{z\phi} = \tfrac{1}{6} r \frac{\partial\Omega}{\partial z} f_1. \tag{51}$$

The last two terms in (50) originate from (41*b*). For axisymmetric dynamo fields, the last term in (50) vanishes, leaving just the ordinary scalar β-term $-\tfrac{1}{3} g_0 \nabla\times\boldsymbol{B}_0$. But the tensor $\boldsymbol{\alpha}$ remains anisotropic, and this is due to the shear. For zero shear ($\partial\Omega/\partial z = 0$ in (51)), $\boldsymbol{\alpha}$ becomes $-\tfrac{1}{3} f_0 \boldsymbol{I}$, hence the ordinary scalar α-term survives. The terms $(:) g_1$ in $\boldsymbol{\alpha}$ stem from the second term in (41*a*).

The extra contribution to $\boldsymbol{\alpha}$ can be estimated by taking $\partial\Omega/\partial z \sim \Omega/R$, $g_1 \sim \tau_c g_0$, $f_1 \sim \tau_c f_0$. For instance, for α_{rr},

$$\frac{\frac{1}{3}\frac{\partial\Omega}{\partial z} g_1}{\frac{1}{3} f_0} \sim \frac{\Omega\tau_c}{R}\frac{\beta}{\alpha}. \tag{52}$$

Here we have written $\tfrac{1}{3} f_0 = \alpha$ and $\tfrac{1}{3} g_0 = \beta$, the usual scalar α- and β-coefficients. For the solar case (52) is of order unity ($\Omega \sim 2.5\times10^{-6}\ \mathrm{s}^{-1}$; $\tau_c \sim 10^5$ s; $R = 7\times10^{10}$ cm; $\alpha \sim 5\ \mathrm{cm\,s^{-1}}$; $\beta \sim 10^{12}\ \mathrm{cm^2\,s^{-1}}$). Likewise, $\alpha_{z\phi} \sim \alpha_{rz} \sim \tfrac{1}{3} f_0$. Hence a large effect is indicated, especially in strongly differentially rotating systems. However, it would be unreasonable to emphasize this point further without allowing the tensors (47 *a*, *b*) to be anisotropic. The main purpose of this example is to illustrate the relative mathematical simplicity ensuing from the use of the displacement-gradient matrix $\boldsymbol{D}$. It is straightforward to replace (42) by the Lagrangian coordinate for a more complicated mean flow, in particular since it need only be defined over a time interval of the order of τ_c, since the correlation functions in (41 a, b) vanish anyway as soon as $\tau \gg \tau_c$.

The example given in this section can also be evaluated on the basis of the work of Rädler (1980, his relation (3.30)), and the result is different. The origin of the difference is that Rädler uses a relation different from (15) to describe the behaviour of $\boldsymbol{B}_0$ during the previous correlation time. This matter is still being investigated.

7. Formal generalization to all orders

This is relevant when the correlation time τ_c is long, i.e. when $\tau_c v/l \sim 1$ rather than $\ll 1$, as is expected to be actually the case. Generalization is straightforward on the basis of the work of Van Kampen (1974). In the basic equation (7) the substitution

$$\boldsymbol{B} = \mathrm{e}^{t\mathbf{R}}\,\boldsymbol{\beta}, \tag{53}$$

is made (note the difference with (9)), so that (7) becomes

$$\frac{\partial}{\partial t}\boldsymbol{\beta} = \mathscr{C}\boldsymbol{\beta}, \quad \mathscr{C}(t) \equiv \mathrm{e}^{-t\mathbf{R}}\,\mathbf{C}(t)\,\mathrm{e}^{t\mathbf{R}}. \tag{54}$$

Van Kampen shows that (54) implies the following equation for $\langle\boldsymbol{\beta}\rangle$:

$$\frac{\partial}{\partial t}\langle\boldsymbol{\beta}\rangle = \mathbf{K}\langle\boldsymbol{\beta}\rangle, \tag{55}$$

with
$$\mathbf{K} = \sum_{n=2}^{\infty}\int_0^{\infty}\mathrm{d}\tau_1\ldots\int_0^{\infty}\mathrm{d}\tau_{n-1}\,\langle\!\langle\!\langle\mathscr{C}_0\mathscr{C}_1\ldots\mathscr{C}_{n-1}\rangle\!\rangle\!\rangle. \tag{56}$$

$\mathscr{C}_i$ is shorthand for

$$\mathscr{C}_i \equiv \mathscr{C}(t-\tau_1-\ldots-\tau_i), \quad \mathscr{C}_0 \equiv \mathscr{C}(t). \tag{57}$$

The triple bracket in (56) denotes the so-called ordered cumulant, the precise definition of which (Van Kampen 1974) is immaterial here, except that it consists of a sum of products of ordinary averages:

$$\langle\!\langle\!\langle\mathscr{C}_0\mathscr{C}_1\ldots\mathscr{C}_{n-1}\rangle\!\rangle\!\rangle = \Sigma\,\langle\mathscr{C}_0\ldots\rangle\,\langle\mathscr{C}_{i_1}\ldots\rangle\ldots\langle\mathscr{C}_{i_k}\ldots\rangle. \tag{58}$$

Apart from the averaging brackets, each summand in (58) contains a permutation of the operators $\mathscr{C}_0\ldots\mathscr{C}_{n-1}$, with $\mathscr{C}_0$ appearing always to the very left.

The dynamo equation follows from (53) and (55):

$$\begin{aligned}\frac{\partial}{\partial t}\boldsymbol{B}_0 &= \frac{\partial}{\partial t}\langle\boldsymbol{B}\rangle = \frac{\partial}{\partial t}\mathrm{e}^{t\mathbf{R}}\langle\boldsymbol{\beta}\rangle\\ &= \mathbf{R}\,\mathrm{e}^{t\mathbf{R}}\langle\boldsymbol{\beta}\rangle + \mathrm{e}^{t\mathbf{R}}\frac{\partial}{\partial t}\langle\boldsymbol{\beta}\rangle\\ &= \mathbf{R}\boldsymbol{B}_0 + \mathrm{e}^{t\mathbf{R}}\,\mathbf{K}\langle\boldsymbol{\beta}\rangle,\end{aligned}$$

or
$$\frac{\partial}{\partial t}\boldsymbol{B}_0 = (\mathbf{R}+\mathrm{e}^{t\mathbf{R}}\,\mathbf{K}\,\mathrm{e}^{-t\mathbf{R}})\,\boldsymbol{B}_0, \tag{59}$$

and it remains to work out the second operator in (59).

Combining (33) and (54), one may relate $\mathscr{C}$ to $\bar{\mathbf{C}}$:

$$\mathscr{C}(t-\tau) = \mathrm{e}^{-(t-\tau)\mathbf{R}}\,\mathbf{C}(t-\tau)\,\mathrm{e}^{(t-\tau)\mathbf{R}} = \mathrm{e}^{-t\mathbf{R}}\,\bar{\mathbf{C}}(\tau)\,\mathrm{e}^{t\mathbf{R}}. \qquad (60)$$

Since τ in (60) is arbitrary, it follows from (57) that

$$\mathscr{C}_i = \mathrm{e}^{-t\mathbf{R}}\,\bar{\mathbf{C}}(\tau_1+\ldots+\tau_i)\,\mathrm{e}^{t\mathbf{R}}. \qquad (61)$$

Consider now the product of n permuted operators $\mathscr{C}_i$:

$$\mathbf{O} \equiv \mathscr{C}_0\mathscr{C}_{j_1}\ldots\mathscr{C}_{j_{n-1}}. \qquad (62)$$

Substitution of (61) in (62) gives

$$\mathbf{O} = \mathrm{e}^{-t\mathbf{R}}\,\bar{\mathbf{C}}(0)\,\bar{\mathbf{C}}(\tau_1+\ldots+\tau_{j_1})\ldots\bar{\mathbf{C}}(\tau_1+\ldots+\tau_{j_{n-1}})\,\mathrm{e}^{t\mathbf{R}}.$$

The presence of internal averaging brackets in $\mathbf{O}$ as in (58) would not interfere with the above reasoning, whence it follows that

$$\langle\!\langle\mathscr{C}_0\ldots\mathscr{C}_{n-1}\rangle\!\rangle = \mathrm{e}^{-t\mathbf{R}}\langle\!\langle\bar{\mathbf{C}}(0)\,\bar{\mathbf{C}}(\tau_1)\ldots\bar{\mathbf{C}}(\tau_1+\ldots+\tau_{n-1})\rangle\!\rangle\,\mathrm{e}^{t\mathbf{R}},$$

and, finally, with (56),

$$\mathrm{e}^{t\mathbf{R}}\,\mathbf{K}\,\mathrm{e}^{-t\mathbf{R}} = \sum_{n=2}^{\infty}\int_0^{\infty}\mathrm{d}\tau_1\ldots\int_0^{\infty}\mathrm{d}\tau_{n-1}\langle\!\langle\bar{\mathbf{C}}(0)\,\bar{\mathbf{C}}(\tau_1)\ldots\bar{\mathbf{C}}(\tau_1+\ldots+\tau_{n-1})\rangle\!\rangle. \qquad (63)$$

The order of magnitude of the nth summand in (63) is $(\tau_c v/l)^n$. For a short correlation time, only the term $n = 2$ contributes:

$$\mathrm{e}^{t\mathbf{R}}\,\mathbf{K}\,\mathrm{e}^{-t\mathbf{R}} \approx \int_0^{\infty}\mathrm{d}\tau_1\langle\!\langle\bar{\mathbf{C}}(0)\,\bar{\mathbf{C}}(\tau_1)\rangle\!\rangle = \int_0^{\infty}\mathrm{d}\tau\,\langle\bar{\mathbf{C}}(0)\,\bar{\mathbf{C}}(\tau)\rangle, \qquad (64)$$

and substitution of (64) in (59) leads back to the FOSA result (35).

If follows from (63) that the dynamo equation (59) contains velocity correlation functions that depend only on $\bar{\boldsymbol{v}}(\tau) = \boldsymbol{D}^{-\tau}\,\boldsymbol{v}(\boldsymbol{r}^{-\tau}, t-\tau)$, in other words, dynamo action at $(\boldsymbol{r}, t)$ depends only on the time history of $\boldsymbol{v}$ measured at *one* material point moving with the mean flow, such that its position is $\boldsymbol{r}$ at time t. This result holds to arbitrary order; by replacing $\boldsymbol{v}$ by $\bar{\boldsymbol{v}}$ the mean flow $\boldsymbol{v}_0$ has been completely accounted for. It is now possible to extend Knobloch's (1978) work to the case $\boldsymbol{v}_0 \neq 0$ by working out the terms $n = 3, 4,$ etc. in (63). However, in the author's view it makes little sense to do so. This is because the convergence of the series (63) is highly questionable when $(\tau_c v/l)$ approaches unity. In other words, just when the terms $n > 2$ in (63) become really important, we should at the same time expect a meaningless result. This was realized by Knobloch (1977, 1978), but his suggestion that the scalar β-coefficient could become negative appears unfounded.

The purpose of (63) is therefore not to provide a series expansion that one could actually evaluate in any useful way; its purpose is rather to show the existence of an expression demonstrating that the dynamo equation depends only on $\bar{\boldsymbol{v}}$ instead of $\boldsymbol{v}$ when $\boldsymbol{v}_0 \neq 0$. One may expect that this property is not affected by the fact that (63) diverges, and carries over to a dynamo theory for long correlation times yet to be developed. A possible, but unproven, implication of (63) is that in setting up a dynamo theory one may ignore the mean flow and in the end replace $\boldsymbol{v}$ by $\bar{\boldsymbol{v}}$ to account for $\boldsymbol{v}_0 \neq 0$.

I am indebted to Drs J. Kuijpers, A. M. Soward, G. A. Stevens and M. Stix for their helpful remarks and constructive criticism on an earlier version of this work. I acknowledge a very illuminating correspondence with Dr K.-H. Rädler.

Appendix A

We wish to prove†

$$\boldsymbol{D}[\boldsymbol{\nabla}^0 \times (\boldsymbol{a}^0 \times \boldsymbol{b}^0)] = \boldsymbol{\nabla} \times [(\boldsymbol{D}\boldsymbol{a}^0) \times (\boldsymbol{D}\boldsymbol{b}^0)], \tag{A 1}$$

with

$$\boldsymbol{a}^0 \equiv \boldsymbol{a}(\boldsymbol{r}^0), \quad \boldsymbol{b}^0 \equiv \boldsymbol{b}(\boldsymbol{r}^0), \quad \boldsymbol{\nabla}^0 = \frac{\partial}{\partial \boldsymbol{r}^0}. \tag{A 2}$$

Here we follow the simplified notation of figure 3: $\boldsymbol{r}^{-\tau} \rightarrow \boldsymbol{r}^0$ and $\boldsymbol{D}^{-\tau} \rightarrow \boldsymbol{D}$. For any matrix $\boldsymbol{D}$

$$\epsilon_{ijk} D_{ip} D_{jq} D_{kr} = \epsilon_{pqr} \det \boldsymbol{D}. \tag{A 3}$$

Multiplying with $D^{-1}{}_{pl}$ and using $\det \boldsymbol{D} = 1$, (A 3) becomes

$$\epsilon_{ljk} D_{jq} D_{kr} = \epsilon_{pqr} D^{-1}{}_{pl}. \tag{A 4}$$

Equation (A 4) implies

$$\begin{aligned}[(\boldsymbol{D}\boldsymbol{a}^0) \times (\boldsymbol{D}\boldsymbol{b}^0)]_l &\equiv \epsilon_{ljk} D_{jq} D_{kr} a^0{}_q b^0{}_r \\ &= \epsilon_{pqr} D^{-1}{}_{pl} a^0{}_q b^0{}_r = [\boldsymbol{D}^{-1*}(\boldsymbol{a}^0 \times \boldsymbol{b}^0)]_l,\end{aligned} \tag{A 5}$$

where * indicates the transposed matrix. The relation between $\boldsymbol{\nabla}$ and $\boldsymbol{\nabla}^0$ follows from

$$\frac{\partial}{\partial x_i} = \frac{\partial x^0{}_j}{\partial x_i} \frac{\partial}{\partial x^0{}_j} = D^{-1}{}_{ji} \frac{\partial}{\partial x^0{}_j}$$

or

$$\boldsymbol{\nabla} = \boldsymbol{D}^{-1*} \boldsymbol{\nabla}^0 = \mathfrak{D}\boldsymbol{\nabla}^0, \quad \text{with } \mathfrak{D} \equiv \boldsymbol{D}^{-1*}. \tag{A 6}$$

Collecting these results and writing $\boldsymbol{c}^0 = \boldsymbol{a}^0 \times \boldsymbol{b}^0$, it remains to prove that

$$\boldsymbol{D}(\boldsymbol{\nabla}^0 \times \boldsymbol{c}^0) = (\mathfrak{D}\boldsymbol{\nabla}^0) \times (\mathfrak{D}\boldsymbol{c}^0). \tag{A 7}$$

Writing down the lth component of the right-hand side of (A 7),

$$\epsilon_{ljk} \mathfrak{D}_{jq} \nabla^0{}_q (\mathfrak{D}_{kr} c^0{}_r) = \epsilon_{ljk} \mathfrak{D}_{jq} \mathfrak{D}_{kr} (\nabla^0{}_q c^0{}_r) + \epsilon_{ljk} c^0{}_r (\nabla_j \mathfrak{D}_{kr}). \tag{A 8}$$

To the first term on the right-hand side we apply (A 4) to find

$$[\mathfrak{D}^{-1*}(\boldsymbol{\nabla}^0 \times \boldsymbol{c}^0)]_l = [\boldsymbol{D}(\boldsymbol{\nabla}^0 \times \boldsymbol{c}^0)]_l.$$

The second term of (A 8) was obtained by applying (A 6) in the form $\mathfrak{D}_{jq} \nabla^0{}_q = \nabla_j$, and it can easily be seen to vanish, because $\nabla_j \mathfrak{D}_{kr}$ is symmetric in the indices j and k. This last point can be shown as follows. $\boldsymbol{D}^{-1} = \partial \boldsymbol{r}^0 / \partial \boldsymbol{r}$, so that $\mathfrak{D}_{kr} = D^{-1}{}_{rk} = \partial x^0{}_r / \partial x_k$. Hence $\nabla_j \mathfrak{D}_{kr} = \partial^2 x^0{}_r / \partial x_j \partial x_k$. This completes the proof of (A 1).

† I am much indebted to Dr K.-H. Rädler, who pointed out to me the short and concise proof given here.

Appendix B. Turbulent transport of a scalar

Scalar transport is governed by

$$\frac{\partial}{\partial t}\rho = -\nabla\cdot\rho V, \quad V = v_0 + v. \tag{B 1}$$

Direct application of the theory (Van Kampen 1976, §12; 1981, chap. 14) gives the following equation for $\rho_0 \equiv \langle\rho\rangle$, see also (14):

$$\begin{aligned}\frac{\partial}{\partial t}\rho_0 = \mathbf{R}\rho_0 + \int_0^\infty \mathrm{d}\tau \, \langle \mathbf{C}(t)\, \mathrm{e}^{\tau\mathbf{R}}\, \mathbf{C}(t-\tau)\, \mathrm{e}^{-\tau\mathbf{R}} \rangle \rho_0, \\ \mathbf{R} \equiv -\nabla\cdot(v_0 \quad) = -v_0\cdot\nabla(\quad), \\ \mathbf{C} \equiv -\nabla\cdot(v \quad) = -v\cdot\nabla(\quad).\end{aligned} \tag{B 2}$$

The latter equations are a consequence of imcompressibility. The analysis of §4 is very easy for scalars. For incompressible flows the relation analogous to (26) is

$$\mathrm{e}^{\tau\mathbf{R}} f(\boldsymbol{r}, t) = f(\boldsymbol{r}^{-\tau}, t). \tag{B 3}$$

Evaluation of $\exp(\tau\mathbf{R})\mathbf{C}(t-\tau)\exp(-\tau\mathbf{R})\rho_0$ is now immediate:

$$\begin{aligned}-\mathrm{e}^{\tau\mathbf{R}}\, \nabla_\sigma v_\sigma(\boldsymbol{r}, t-\tau)\, \mathrm{e}^{-\tau\mathbf{R}}\, \rho_0(\boldsymbol{r}, t) &= -\mathrm{e}^{\tau\mathbf{R}}\, v_\sigma(\boldsymbol{r}, t-\tau)\, \nabla_\sigma \rho_0(\boldsymbol{r}^\tau, t) \\ &= -v_\sigma(\boldsymbol{r}^{-\tau}, t-\tau)\, \nabla^{-\tau}{}_\sigma \rho_\pi(\boldsymbol{r}, t) \\ &= -v_\sigma(\boldsymbol{r}^{-\tau}, t-\tau)\, (\nabla_\nu \rho_0) \frac{\partial x_\nu}{\partial x^{-\tau}{}_\sigma} \\ &= -(\boldsymbol{D}^{-\tau} v(\boldsymbol{r}^{-\tau}, t-\tau))\cdot\nabla\rho_0 = -\bar{v}(\tau)\cdot\nabla\rho_0.\end{aligned}$$

Hence (B 2) can be written as

$$\begin{aligned}\frac{\partial}{\partial t}\rho_0 + \nabla\cdot\rho_0 v_0 = \nabla\cdot\boldsymbol{G}\cdot\nabla\rho_0, \\ \boldsymbol{G} \equiv \int_0^\infty \mathrm{d}\tau \, \langle \bar{v}(0)\, \bar{v}(\tau) \rangle, \quad \rho_0 = \langle\rho\rangle.\end{aligned} \tag{B 4}$$

Generalization to a long correlation time along the lines of §7 is straightforward.

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